A Duel between Duals of Polygons

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Given a polygon in the plane, there are three associated constructions which yield fresh polygons, two of which are *duals* in the sense that the dual of the dual is the original, and one for which this is not actually the case.

The three constructions are presented, with ‘results’ that the reader should verify for themselves, because it really helps to develop facility in passing from one prsentation of a polygon to another.

The ‘duel’ of the title refers to what the dual affords in the way of ease of proof of results.

# Combinatorial Dual

#### Construction

Given a polygon *P*, take the midpoints of the edges of *P* as the vertices of *CD*(*P*).

The result is a new polygon.

#### Result CD1

The dual is similar to the original if and only if *P* is equilateral.

This construction has been studied at length (ref). When applied to quadrilaterals and iterated, the quadrilaterals formed tend to a parallelogram.

An apparently more general construction is to choose a non-zero number µ and to mark points which divide the edges in the ratio of µ : 1 – µ in both directions. For each vertex of *P*, join the points that divide the edges incident with *P* in the ratio µ : 1 – µ.

This construction is equivalent to truncating the polygon at the vertices in a uniform manner.

#### Result CD2

The midpoint construction occurs when µ = 1/2.

The result of the µ construction is the midpoint construction scaled by a factor σ where σ = ???

# The Projective Dual

The projective dual has a geometric interpretation as well as an algebraic procedure, but first it is necessary to embed the Cartesian plane in the projective plane, by adjoining a line at infinity made up of points, each of which corresponds to a family of parallel lines in the plane.

A manifestation of the projective dual has been deeply studied (see for example ref) especially in relation to conics, where given a conic, poles and polars correspond so that a presentation of the dual figure is provided along with the figure itself.

## The Projective Plane

Geometrically, imagine your eye (as a point) looking down on a plane in which lies the polygon *P*. The vertices of *P* correspond to rays of light, or lines, through your eye. A line in the plane corresponds to all the rays through your eye to points on the line, which is a plane through your eye. Treat your eye as the origin of a vector space. Lines through your eye are ‘Ppoints’ and planes through your eye are ‘Plines’.

Algebraically this can be achieved by converting Cartesian coordinates [*a*, *b*] for a point into homogeneous coordinates {λ[*a, b*, 1] : λ in *R*} to describe the line through the origin. Since attention is on the lines as a direction rather than as made up of vectors, it is sufficient to use a single representative to refer to it. So the homogeneous coordinates [*a*, *b*, *c*] are equivalent to [μ*a*, μ*b*, μ*c*] for any non-zero μ.

A different projection of a configuration in the Caretsian plane can be obtained by forming the projective coordinates and then choosing a different plane (other than the plane *z* = 0) on which to project. For example, to project onto the plane *y* = 0, divide homogeneous coordinates by the *y*-coordinate and then delete the *y*-coordinate;

#### Result PP1

To project onto the plane α*x* + β*y* + γ*z* = 1, scale the homogeneous coordinates of a point so that the linear combination is 1, then eliminate any one of the coordinates (since it can be recovered from others);

Find the function being plotted on each of the planes *y* = 0, *x* = 0, and *x* + *y* + *z* = 1 of the function [*x*, sin(*x*), 0] on the plane *z* = 0.

A line in the Cartesian plane corresponds to a plane through the eye. The equation of a plane through the eye is achieved by homogenising the equation of the line in the plane. Thus *ax* + *by* + *c* = 0 becomes the plane *ax* + *by* + *cz* = 0.

#### Result PD1

The line *a(x* – *p*) + *b*(*y* – *q*) + *c* = 0 in the Cartesian plane passes through the point [*p, q*] and has projective label [*a*, *b*, *ap* + *bq* + *c*].

The line through the point [*p,* *q*] in the Cartesian plane with slope *m* has the Cartesian equation *mx* – *y* + *q* – *mp* and so has projective label [*m*, –1, *q* – *mp*].

## Projective Dual

The construction is very simple: since points and lines both have triples of real numbers as their homogeneous (equivalence class of) coordinates, simple interchange points and lines. This means declaring the homogeneous coordinates of a point to be the homogeneous coordinates of a Dual-Point, namely the Dual-Line. Note, Cinderella, a dynamic geometry programme, actually uses homogeneous coordinates, converting these to Cartesian upon request, so the duality is easily achieved: simple tell the programme whether a triple is ti be considered as a point or a line.

#### Construction

Interchange points and lines

### Results

There are a number of questions which arise, especially when you want to present polygons and duals in dynamic geometry software.

#### Result PD1

The line through two points in homogeneous coordinates converts to the point of intersection of the lines dual to the two points.

The point of intersection of two lines in homogeneous coordinates converts to the line joining the points dual to the two lines.

#### Result PD2

To enable the user to move polygons around easily, it is necessary to present a polygon relative to a point, so that the polygon can be translated easily in the plane, and likewise its dual.

Given a polygon in the plane, presented in terms of a list of homogeneous coordinates of its vertices, relate the homogeneous coordinates of points relative to some displayed and moveable point *P*0.

Present the vertices of the dual polygon relative to the point *Q*0.

Present a line in homogeneous coordinates relative to a point *P*0.

Present the dual point

The cross-product of two homogenous triples is the homogeneous triple for the intersection of the corresponding lines, or the line joining the corresponding points. Think geometrically.

The cross product of the translation of two points is the translation of the cross product by the cross product of their difference with the translation.

Therefore

## NOTES

Projective duals: p is incident with line L iff pL=0 as inner product (L written as column vectors

Line through p and q is p X q; p, q, r are collinear when det(p, q, r) = 0

L meets M at point L X M; L, M, N are coincident when det = 0

Polarity: p perp q rel to A if pAq(T)=0 where A is symmetric. Aq(T) is a line thru p and q is a point on line thru p

So, pole-polar duality wrt a conic is vector-space duality; the symmetric matrix A corresponds to a conic, and projectively, the plane perp to the vector p is presented as the polar of the pole represents by p

Appolonius knew about polarities wrt a conic via the tangent construction

# Point-Based Polygon Duals

Imagine a polygon in the Cartesian plane, and a point *P*0 somewhere in the plane. Imagine a point *P* traversing the polygon and a unit vector from *P*0 and in the direction of the current edge. Refer to the unit vector as the unit-sweep. When *P* gets to a vertex, the unit-sweep vector rotates through the exterior angle of the polygon until it is in line with the next edge.

While *P* is traversing an edge, the vector *PP*0 changes its direction. While the unit-sweep vector is rotating at a vertex, the vector *PP*0 is stationary, but the angle of the unit-sweep vector is changing.

Think of the movement of *P* as determined by a parameter which traverses a line segment. The line segment is divided into 2*N* consecutive segments of equal length. As a parameter point *Pp* traverses an odd-numbered segment, the point *P* is driven along the corresponding edge, and when *Pp* traverses an even-numbered segment, the unit-sweep vector rotates through the exterior angle.

The motion of the unit-sweep vector is thus determined by two sets of angles.

The point-based dual of the polygon with respect to the point *P*0 is formed by interchanging these two sets of angles, relative to some base point *Q*0.

#### Result PBD1

Given the Cartesian coordinates of the vertices of a polygon *P* in the plane, and a base-point *P*0, determine the Cartesian coordinates of the *P*0-based dual of *P*.

For a given position of a parameter point *Pp* on an odd-numbered segment of the parameter unit interval, determine the position of the corresponding point *P* on the polygon’s corresponding edge.

For a given position of a parameter point *Pp* on an even-numbered segment of the parameter unit interval, determine the direction of the corresponding unit-sweep vector at polygon’s corresponding vertex.

Comment

The third task is not quite as easy as it first appears, because you need to determine angles, and this in turn involves inverse tangents, which are specified in the interval (-180°, 180]. Comparing such angles requires considerable care!

the vectors *PP*0 rotate as *P* traverses an edge.